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Determination of a two-dimensional heat source: Uniqueness, regularization and error estimate

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Abstract

Let Q be a heat conduction body and let $\varphi = \varphi(t)$ be given. We consider the problem of finding a two-dimensional heat source having the form $\varphi(t)f(x, y)$ in Q. The problem is ill-posed. Assuming ∂Q is insulated and $\varphi \not\equiv 0$, we show that the heat source is defined uniquely by the temperature history on ∂Q and the temperature distribution in Q at the initial time $t = 0$ and at the final time $t = 1$. Using the method of truncated integration and the Fourier transform, we construct regularized solutions and derive explicitly error estimate.

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1. Introduction

Let Q be a heat conduction body having a constant conductivity, assumed to be equal to 1, and having an insulated boundary. In this paper, we consider the problem of identifying a heat source in the inside of Q from the temperature history on a part of ∂Q and the temperature distribution in Q at the initial time $t = 0$ and at the final time $t = 1$, say. In other words, the problem has Cauchy data on a part of the boundary.

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The problem has been studied intensively for the last three decades (see, e.g., [6,10,11,13,14,16]). Letting u be the temperature in Q and $F = F(x, y, t, u)$ be the heat source, one has the equation

$$-\frac{\partial u}{\partial t} + \Delta u = F(x, y, t, u).$$

The problem is severely ill-posed. In fact, because of the extreme sensitiveness to measurement errors (see, e.g., [3]), the inverse heat source problem is difficult. Measured data is usually the result of discretely experimental measurements and, of course, is subject to error. Hence, a solution corresponding to the data does not always exist and moreover, solutions, even thought they exist, do not depend continuously on the given data. This, of course, makes a numerical treatment impossible. Thus, one has to resort to a regularization.

To simplify the problem, many preassumptions on the form of the heat source had been given. Roughly speaking, we can approximate the function $F(x, y, t, u)$ by a function which has the form

$$\sum_{n=0}^{N} \Psi_n(u)\varphi_n(t) f_n(x, y).$$

The first order of the latter form can be written as

$$F \approx \Psi_0(u) + \varphi_0(t) + f_0(x, y) + \varphi_1(t) f_1(x, y) + \cdots .$$

In [10,11], the author assumed that

$$F(x, y, t, u) = g_0(x, t) + f_1(x)g_1(t) + f_2(t)g_2(x),$$

where f_1, f_2 are unknown. In [13],

$$F(x, y, t, u) = f(u) + r(x, t),$$

where f is unknown (see also [12] for a similar form of the heat source). In [4,5,14,17], we get the separated form

$$F(x, y, t, u) = \sigma(t) f(x)$$

in which one of two functions is unknown.

Now, in the present paper, for simplicity, as in [6], we shall consider a model in which the heat source has the separated form $\varphi(t) f(x, y)$ where f is unknown. As we shall see, with minimum smoothness assumption, the given Cauchy data is sufficient to the uniqueness of solution. However, as discussed the problem is still ill-posed.

Precisely, we assume that Q is represented by the square $(0, 1) \times (0, 1)$. Letting $u = u(x, y, t)$ be the temperature in Q, we consider the problem of identifying a pair of functions (u, f) satisfying

$$-\frac{\partial u}{\partial t} + \Delta u = \varphi(t) f(x, y) \tag{1}$$

for $(x, y, t) \in Q \times (0, 1)$.

Since the boundary of Q is insulated, we have

$$\begin{cases} u_x(0, y, t) = u_x(1, y, t) = 0, \\ u_y(x, 0, t) = u_y(x, 1, t) = 0, \end{cases} \quad (x, y, t) \in Q \times (0, 1). \tag{2}$$

Finally, we have the temperature history on a part of ∂Q

$$u(1, y, t) = u(x, 1, t) = 0 \quad (x, y, t) \in Q \times (0, 1) \tag{3}$$

and the temperature distribution in Q at $t = 0$ and 1

$$u(x, y, 0) = 0; \quad u(x, y, 1) = g(x, y) \quad (x, y) \in Q. \tag{4}$$

Here, φ, g are given functions. In (3) and (4) the conditions, in which $u(1, y, t)$, $u(x, 1, t)$, $u(x, y, 0)$ are vanished, are simplified. In fact, the method of our paper can be thoroughly applied to the problem associated with more general data. Hence, to simplify computation and to point out clearly main ideas of the method, we only consider the simplified conditions as in (3) and (4).

Our problem is equivalent to the one of finding the function $f = f(x, y)$ satisfying a Volterra equation of the first kind (see, e.g., [8])

$$K_\varphi f(x, y) = -g(x, y) \tag{5}$$

for $(x, y) \in Q$, where

$$N(x, y, t; \xi, \eta, \tau) = \frac{1}{4\pi(t-\tau)} \exp\left(-\frac{(y-\eta)^2}{4(t-\tau)}\right)\left[\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) + \exp\left(-\frac{(x+\xi)^2}{4(t-\tau)}\right)\right],$$

$$K_\varphi f(x, y) = \int_0^1 \int_0^1 \int_0^1 N(x, y, 1; \xi, \eta, \tau)\varphi(\tau) f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\tau.$$

We note at once that the problem of existence of a solution is not considered here. The set of the (φ, g)'s for which system (1)–(4) has no solution is dense in $L^2(0, 1) \times L^2(Q)$. Indeed, from (5) we can show that g is smooth if $f \in L^2(Q)$, $\varphi \in L^2(0, 1)$. However, in practice, g and φ come from experimental measurements and thus given as finite sets of points that are conveniently patched into L^2-functions. Hence g is in general non-smooth. With these non-exact data, system (1)–(4) usually has no solution. Thus, as mentioned, we deal with a problem that possibly has not a solution and hence we would have to resort to a regularization. If we denote by (g_0, φ_0) the (probably unknown) exact data corresponding to an exact solution (u_0, f_0) of system (1)–(4) then, from the (known) non-exact data (g, φ) approximating (g_0, φ_0), we shall construct regularized solutions of (1)–(4).

As we have shown in (5), the integral operator K_φ depends on the non-exact data φ which implies that K_φ is also non-exact. Hence, it is difficult to derive error estimate of a regularization method. In the case of one spatial dimension, the linear integral equation has been treated widely in the past few decades (see, e.g., [1,2,7,9,15] and our recent paper [6]). Although the mathematical literature in the one-dimensional case is rather impressive, it is quite scarce in the two dimensional case. In the present paper, using a variational form of (1)–(4) we shall transform the problem to the one of finding f from its Fourier transform calculated from the given data (g, φ) (see Lemma 1). Assuming the discrepancy between the exact data (g_0, φ_0) and the non-exact data (g, φ) is of order $\varepsilon > 0$, we shall use a new method of truncated integration to construct (from the non-exact data (g, φ)) a regularized solution f_ε. Moreover, the error between f_ε and the exact solution f_0 (which depends on the properties of φ_0 and the smoothness of f_0) will be given explicitly. The remainder of the paper is divided into two sections. In Section 2, we shall set some notations and state main results of our paper. In Section 3, we give the proofs of the main results.

2. Notations and main results

We recall that $Q=(0,1)\times(0,1)$. We denote by $C([0,1],H^1(Q))$ the set of continuous functions $u(.,t):[0,1]\to H^1(Q)$ and by $C^1([0,1],L^2(Q))$ the set of C^1-functions $u(.,t):[0,1]\to L^2(Q)$. From now on, we shall assume that $g\in L^2(Q)$ and $\varphi\in L^2(0,1)$. We also denote by $c(.)$, $s(.)$, respectively, the functions cos(.), sin(.) for short. From (1) to (2) we get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle u,\psi\rangle-\langle u_x,\psi_x\rangle-\langle u_y,\psi_y\rangle=\varphi\langle f,\psi\rangle \quad \forall\psi\in H^1(Q), \tag{6}$$

where $\langle\,,\,\rangle$ is the inner product in $L^2(Q)$.

Using (6), we first have the following lemma.

Lemma 1. *If* $u\in C^1([0,1];L^2(Q))\cap C([0,1];H^1(Q))$, $f\in L^2(Q)$ *satisfy* (3), (4), (6) *then, for all* $\alpha,\beta\in\mathbb{C}$,

$$\begin{aligned}&\mathrm{e}^{\alpha^2+\beta^2}\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\\&\quad=-\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t\int_0^1\int_0^1 f(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{7}$$

We note that, it may undergo non-trivial changes when applied (5) to get an equation as (7).

From Lemma 1, we have informally

$$\int_0^1\int_0^1 f(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y=-\frac{\mathrm{e}^{\alpha^2+\beta^2}\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t}.$$

Hence, the function *f* is possibly identified if the set

$$\left\{(\alpha,\beta)\in\mathbb{R}^2\,\middle|\,\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t=0\right\}$$

is negligible in an appropriate sense. In fact, from the properties of analytic functions, we can show that, if $\varphi\not\equiv0$, the set is a union of countable concentric circles and, hence, the Lebesgue measure of the set is zero.

Moreover, we have

Lemma 2. *Let* $\varphi_0\in L^2(0,1)$ *and* $R>0$. *Put*

$$\Phi_0(\mu)=\int_0^1\mathrm{e}^{t\mu^2}\varphi_0(t)\,\mathrm{d}t$$

$$B_R=\left\{(\alpha,\beta)\,\middle|\,\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)<R\right\}$$

$$v_0(t)=\int_1^t\varphi_0(s)\,\mathrm{d}s.$$

If there exists a $\delta \in (0, 1)$ *such that* v_0 *is non-positive (or non-negative) on* $[\delta, 1]$ *and that* $v_0 \not\equiv 0$ *on* $[\delta, 1]$ *then we can find constants* $\gamma, R_0 \in (0, 1)$, $C_0 > 0$ *independent of R such that*

$$m(B_R) \leqslant C_0 R^{\gamma} \quad \text{for all } 0 < R \leqslant R_0,$$

where $m(B_R)$ *is the Lebesgue measure of* B_R.

The set of the function φ_0 satisfying Lemma 2 is very large. For example, if

$$\varphi_0(t) = (1-t)^m (a + (1-t)\psi(t)), m \text{ integer} \geqslant 0 \quad \text{with } a \neq 0, \psi \in L^2(0, 1),$$

then v_0 satisfies the condition of Lemma 2. Now, we have the uniqueness result.

Theorem 1. *Let* $u_1, u_2 \in C^1([0, 1]; L^2(Q)) \cap C([0, 1]; H^1(Q))$ *and* $f_1, f_2 \in L^2(Q)$.

If u_i, f_i *satisfy* (1)–(4) $(i = 1, 2)$ *and* $\varphi \not\equiv 0$, *then*

$$(u_1, f_1) = (u_2, f_2).$$

We give two regularization results.

Theorem 2. *Let* $C_0, \varepsilon > 0$ *and let* $\varphi_0 \in L^2(0, 1)$, $g_0 \in L^2(Q)$ *and* (u_0, f_0) *be the exact solution of* (1)–(4) *corresponding* (φ_0, g_0) *in right-hand side. Assume that* φ, g *satisfy*

$$\|\varphi - \varphi_0\|_{L^2(0,1)} < \varepsilon, \|g - g_0\|_{L^2(Q)} < \varepsilon \tag{8}$$

and $\varphi(t) > C_0$, $\varphi_0(t) > C_0$ *a.e. on* $(0, 1)$.

Then from g, φ *we can construct a regularized solution* f_ε, *such that*

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f_0\|_{L^2(Q)} = 0. \tag{9}$$

If we assume, in addition, that $f_0 \in H^2(Q)$, *then*

$$\|f_\varepsilon - f_0\|_{L^2(Q)} \leqslant 4\sqrt{\frac{\pi e^3}{C_0^4}(C_0 + \|g_0\|_{L^2(Q)} + 1)^2 \varepsilon + \frac{384}{\pi^2}\|f_0\|^2_{H^2(Q)} \varepsilon^{1/8}}.$$

Theorem 3. *Let* $\varepsilon > 0$, $a \in (0, \frac{1}{2})$ *and let assumptions of Lemma* 2 *hold. If* g, φ *are the measured data satisfying*

$$\|\varphi - \varphi_0\|_{L^2(0,1)} < \varepsilon \quad \text{and} \quad \|g - g_0\|_{L^2(Q)} < \varepsilon$$

then, for each $a \in (0, \frac{1}{2})$, *we can construct from* g, φ *a function* f_ε *such that*

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f_0\|_{L^2(Q)} = 0.$$

Moreover, if $f_0 \in H^2(Q)$ *then for each* $b \in \left(\max\left\{0, \frac{1-4a}{4}\right\}, \frac{1-2a}{2}\right)$, *we can find* $C_b > 0, \gamma_b > 0$ *independent of* $f_0, g_0, \varphi_0, \varepsilon$ *such that*

$$\|f_\varepsilon - f_0\|_{L^2(Q)} \leqslant \sqrt{C_b \varepsilon^{\gamma_b} + \frac{384}{\pi^2}\|f_0\|^2_{H^2(Q)}\left(b \ln \frac{1}{\varepsilon}\right)^{-1/2}}.$$

3. Proofs

Proof of Lemma 1. Choosing $\psi(x, y) = c(\alpha x)c(\beta y)$ in (6), we get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 u(x, y, t)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y + \alpha\int_0^1\int_0^1 u_x(x, y, t)s(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y$$
$$+\beta\int_0^1\int_0^1 u_y(x, y, t)c(\alpha x)s(\beta y)\,\mathrm{d}x\,\mathrm{d}y = \varphi(t)\int_0^1\int_0^1 f(x, y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y. \tag{10}$$

Since $u(1, y, t) = 0$, we have

$$\int_0^1\int_0^1 u_x(x, y, t)s(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y = -\alpha\int_0^1\int_0^1 u(x, y, t)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

Similarly,

$$\int_0^1\int_0^1 u_y(x, y, t)c(\alpha x)s(\beta y)\,\mathrm{d}x\,\mathrm{d}y = -\beta\int_0^1\int_0^1 u(x, y, t)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

Substituting (11), (12) into (10) gives

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 u(x, y, t)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_0^1\int_0^1 [(\alpha^2+\beta^2)u(x, y, t) + \varphi(t)f(x, y)]c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y. \tag{13}$$

Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{(\alpha^2+\beta^2)t}\int_0^1\int_0^1 u(x, y, t)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\right)$$
$$= -\varphi(t)\mathrm{e}^{(\alpha^2+\beta^2)t}\int_0^1\int_0^1 f(x, y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y,$$

we get

$$\mathrm{e}^{\alpha^2+\beta^2}\int_0^1\int_0^1 g(x, y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= -\int_0^1 \varphi(t)\mathrm{e}^{(\alpha^2+\beta^2)t}\,\mathrm{d}t\int_0^1\int_0^1 f(x, y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y.$$

This completes the proof of Lemma 1. □

Proof of Lemma 2. We shall prove $|\Phi_0(\mu)| \to +\infty$ as $\mu \to \pm\infty$. We have

$$\Phi_0(\mu) = \int_0^1 e^{t\mu^2}\varphi_0(t)\,dt = \int_0^1 e^{t\mu^2} v_0'(t)\,dt$$
$$= e^{t\mu^2} v_0(t)|_0^1 - \mu^2 \int_0^1 v_0(t)e^{t\mu^2}\,dt = -v_0(0) - \mu^2 \int_0^1 v_0(t)e^{t\mu^2}\,dt.$$

By the properties of v_0 there exist t_0 and $\varepsilon_1 > 0$ such that

$$\delta < t_0 - \varepsilon_1 < t_0 < t_0 + \varepsilon_1 < 1$$

and

$$v_0(t) < 0 \quad \forall t \in [t_0 - \varepsilon_1, t_0 + \varepsilon_1].$$

We put

$$C_1 \equiv \min_{[0,t_0-\varepsilon_1]}(-v_0(t)), \qquad C_2 \equiv \min_{[t_0-\varepsilon_1,t_0+\varepsilon_1]}(-v_0(t)) > 0.$$

We have

$$\Phi_0(\mu) = -v_0(0) - \left[\int_0^{t_0-\varepsilon_1} + \int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} + \int_{t_0+\varepsilon_1}^1 v_0(t)e^{t\mu^2}\,dt\right]\mu^2$$
$$\geqslant -v_0(0) + \left[C_1\int_0^{t_0-\varepsilon_1} e^{t\mu^2}\,dt + C_2\int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} e^{t\mu^2}\,dt\right]\mu^2$$
$$\geqslant -v_0(0) + [C_1(e^{(t_0-\varepsilon_1)\mu^2} - 1) + C_2(e^{(t_0+\varepsilon_1)\mu^2} - e^{(t_0-\varepsilon_1)\mu^2})]$$
$$\geqslant -v_0(0) + C_1 e^{(t_0+\varepsilon_1)\mu^2}(e^{-2\varepsilon_1\mu^2} - e^{-(t_0+\varepsilon_1)\mu^2}) + C_2 e^{(t_0+\varepsilon_1)\mu^2}(1 - e^{-2\varepsilon_1\mu^2}).$$

Therefore, $|\Phi_0(\mu)| \to +\infty$ as $\mu \to \pm\infty$. Hence, by the analyticity of $\Phi_0(\mu)$, it follows that $\Phi_0(\mu)$ has only finite zeros μ_j, $j = 1, \ldots, p$ on the real axis. We can write

$$\Phi_0(\mu) = \Phi_1(\mu)\prod_{j=1}^{p}(\mu - \mu_j)^{m_j},$$

where $m_j = 1, 2, \ldots\ \forall j = \overline{1, p}$, $|\Phi_1(\mu)| \to +\infty$ as $\mu \to \pm\infty$ and $\Phi_1(\mu) \neq 0$ for every $\mu \in \mathbb{R}$. By $|\Phi_1(\mu)| \to +\infty$ as $\mu \to \pm\infty$ and $\Phi_1(\mu) \neq 0$ for every μ, there exists $C_3 > 0$ such that $|\Phi_1(\mu)| \geqslant C_3$ for every μ. Hence,

$$\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right| \geqslant C_3 \prod_{j=1}^{p}\left|\sqrt{\alpha^2+\beta^2} - \mu_j\right|^{m_j},$$

where $m_j = 1, 2, \ldots.$

Without loss of generality, we can assume that $0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_p$ (if $\mu_j < 0$ then $|\sqrt{\alpha^2+\beta^2} - \mu_j| \geqslant |\mu_j|$).

Putting $d = \min_{1\leqslant s\leqslant p-1}(\mu_{s+1} - \mu_s)$, $M = \sum_{s=1}^p m_s$ and $\mu = \sqrt{\alpha^2+\beta^2} \geqslant 0$.

Choose $R_0 = \min\{C_3 d^M, \frac{1}{2}, C_3 (\frac{\mu_1}{2})^{2m_1} d^{M-2m_1}\}$ and $\delta_s = \frac{R^{1/2m_s}}{C_3^{1/2m_s} d^{(M-2m_s)/2m_s}}$, $1 \leqslant s \leqslant p$ we then have

(i) If $\mu_s + \delta_s \leqslant \mu \leqslant \mu_{s+1} - \delta_{s+1}$, $s = \overline{1, p-1}$, we have

$$|\Phi_0(\mu)| \geqslant C_3 \prod_{j=1}^p \left|\sqrt{\alpha^2+\beta^2} - \mu_j\right|^{m_j} \geqslant C_3 \delta_s^{m_s} \delta_{s+1}^{m_{s+1}} d^{M_s} = R,$$

where $M_s = M - m_s - m_{s+1}$.

(ii) If $\mu_p + \delta_p < \mu$, we have $|\Phi_0(\mu)| \geqslant C_3 d^{M-m_p} \delta_p^{m_p} = \beta$ and the choice $\delta_p = d\left(\frac{R}{d^M C_3}\right)^{(1/2m_p)}$ with $R \leqslant d^M C_3$ involves that $\beta \geqslant R$.

If $0 \leqslant \mu < \mu_1 - \delta_1$ the same proof as before with $\delta_1 = d\left(\frac{R}{d^M C_3}\right)^{(1/2m_1)}$ implies that $|\Phi_0(\mu)| \geqslant R$.

In the case where $\mu_1 = 0$, i.e., the roots μ_i are such that $0 = \mu_1 < \mu_2 < \cdots < \mu_p$ the previous proof is still valid with $R_0 = \min\{C_3 d^M, \frac{1}{2}\}$.

Which means

$$B_R \subset \bigcup_{s=1}^p \left\{(\alpha,\beta)/\mu_s - \delta_s < \sqrt{\alpha^2+\beta^2} < \mu_s + \delta_s\right\}.$$

Hence

$$m(B_R) \leqslant \sum_{s=1}^p 4\pi\mu_s\delta_s \leqslant 4\pi \max_{s=\overline{1,p}} \mu_s \sum_{s=1}^p \delta_s = 4\pi d \max_{s=\overline{1,p}} \mu_s \sum_{s=1}^p \frac{R^{1/2m_s}}{C_3^{1/2m_s} d^{M/2m_s}}.$$

Choosing $\gamma = \min_{1\leqslant s\leqslant p}\left\{\frac{1}{2m_s}\right\}$, we complete the proof of Lemma 2. □

Proof of Theorem 1. Put $v = u_1 - u_2$ and $f = f_1 - f_2$, then v and f satisfy (6) and

$$v(x,y,1) = 0. \tag{14}$$

Put

$$\tilde{f}(x,y) \equiv \frac{1}{4}\begin{cases} f(x,y) & (x,y) \in (0,1)\times(0,1), \\ f(-x,-y) & (x,y) \in (-1,0)\times(-1,0), \\ f(-x,y) & (x,y) \in (-1,0)\times(0,1), \\ f(x,-y) & (x,y) \in (0,1)\times(-1,0), \\ 0 & (x,y) \notin (-1,1)\times(-1,1). \end{cases}$$

Then

$$\widehat{\tilde{f}}(\alpha,\beta)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\tilde{f}(x,y)\mathrm{e}^{-\mathrm{i}(x\alpha+y\beta)}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{2\pi}\int_0^1\int_0^1 f(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y. \tag{15}$$

By (7) and (15), we get

$$\left[\int_0^1 \mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t\right]\widehat{\tilde{f}}(\alpha,\beta)=0. \tag{16}$$

Let

$$h(\alpha,\beta)\equiv\int_0^1 \mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t=\sum_{n=0}^{\infty}\frac{(\alpha^2+\beta^2)^n}{n!}\int_0^1\varphi(t)t^n\,\mathrm{d}t.$$

If $h\equiv 0$, by Weierstrass theorem, we have $\varphi\equiv 0$, which is a contradiction. Hence $h\not\equiv 0$. This implies that there exist $(\alpha_0,\beta_0)\in\mathbb{C}\times\mathbb{C}$ and $r>0$ such that $|h(\alpha,\beta)|>0$ for every $(\alpha,\beta)\in B((\alpha_0,\beta_0),r)$ where $B((\alpha_0,\beta_0),r)$ is the ball with center at (α_0,β_0) and radius r.

Therefore

$$\widehat{\tilde{f}}(\alpha,\beta)=0\quad\forall(\alpha,\beta)\in B.$$

Since $\widehat{\tilde{f}}$ is an entire function, we get

$$\widehat{\tilde{f}}(\alpha,\beta)=0\quad\forall(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}.$$

Hence $\tilde{f}=0$ a.e.

This follows $f=0$ a.e. on Q which involves $v=0$ a.e. on Q since the variational problem (6) has a unique solution.

This completes the proof of Theorem 1. □

Before proving two main regularization results, we state and prove a necessary lemma. Note that it is of independent interest.

Lemma 3. *Let $D_r=\{(\alpha,\beta)/\alpha^2+\beta^2\leqslant r^2\}$ and $r>\sqrt{2}$. If $f_0\in H^2(Q)$ then*

$$\int_{\mathbb{R}^2\setminus D_r}\left[\int_Q f_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\right]^2\mathrm{d}\alpha\,\mathrm{d}\beta\leqslant 1536\|f_0\|^2_{H^2(Q)}r^{-1}.$$

Proof of Lemma 3. From the denseness of $C^\infty(\overline{Q})$ in $H^2(Q)$, we only consider the case $f_0 \in C^\infty(\overline{Q})$.
We have

$$\begin{aligned}
&\int_0^1\int_0^1 f_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}y\,\mathrm{d}x\\
&= -\int_0^1\int_0^1 \frac{\partial f_0}{\partial x}(x,y)\frac{s(\alpha x)}{\alpha}c(\beta y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^1\left[f_0(x,y)\frac{s(\alpha x)}{\alpha}\right]_{x=0}^{x=1} c(\beta y)\,\mathrm{d}y\\
&= -\int_0^1\int_0^1 \frac{\partial f_0}{\partial x}(x,y)\frac{s(\alpha x)}{\alpha}c(\beta y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 f_0(1,y)\frac{s(\alpha)}{\alpha}c(\beta y)\,\mathrm{d}y\\
&= \int_0^1\int_0^1 \frac{\partial^2 f_0}{\partial x\,\partial y}(x,y)\frac{s(\beta y)}{\beta}\frac{s(\alpha x)}{\alpha}\,\mathrm{d}x\,\mathrm{d}y - \int_0^1\left[\frac{\partial f_0}{\partial x}(x,y)\frac{s(\beta y)}{\beta}\right]_{y=0}^{y=1}\frac{s(\alpha x)}{\alpha}\,\mathrm{d}x\\
&\quad - \int_0^1 \frac{\partial f_0}{\partial y}(1,y)\frac{s(\beta y)}{\beta}\frac{s(\alpha)}{\alpha}\,\mathrm{d}y + \left[f_0(1,y)\frac{s(\beta y)}{\beta}\right]_{y=0}^{y=1}\frac{s(\alpha)}{\alpha}\\
&= \int_0^1\int_0^1 \frac{\partial^2 f_0}{\partial x\,\partial y}(x,y)\frac{s(\beta y)}{\beta}\frac{s(\alpha x)}{\alpha}\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 \frac{\partial f_0}{\partial x}(x,1)\frac{s(\beta)}{\beta}\frac{s(\alpha x)}{\alpha}\,\mathrm{d}x\\
&\quad - \int_0^1 \frac{\partial f_0}{\partial y}(1,y)\frac{s(\beta y)}{\beta}\frac{s(\alpha)}{\alpha}\,\mathrm{d}y + f_0(1,1)\frac{s(\beta)}{\beta}\frac{s(\alpha)}{\alpha}.
\end{aligned}$$

We have

$$\left|\int_0^1\int_0^1 \frac{\partial^2 f_0}{\partial x\,\partial y}(x,y)\frac{s(\beta y)}{\beta}\frac{s(\alpha x)}{\alpha}\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant k(\beta)k(\alpha)\left\|\frac{\partial^2 f_0}{\partial x\,\partial y}\right\|_{L^2(Q)},$$

where

$$k(\theta) = \begin{cases} 1, & |\theta| \leqslant 1,\\ \frac{1}{|\theta|}, & |\theta| > 1.\end{cases}$$

We have

$$\begin{aligned}
f_0(1,1) &= \int_0^1\int_0^1 \frac{\partial^2(xyf_0)}{\partial x\,\partial y}\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_0^1\int_0^1 \left(f_0(x,y) + y\frac{\partial f_0}{\partial y}(x,y) + x\frac{\partial f_0(x,y)}{\partial x} + \frac{\partial^2 f_0(x,y)}{\partial x\,\partial y}xy\right)\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

hence

$$|f_0(1,1)| \leqslant \|f_0\|_{L^2(Q)} + \left\|\frac{\partial f_0}{\partial y}\right\|_{L^2(Q)} + \left\|\frac{\partial f_0}{\partial x}\right\|_{L^2(Q)} + \left\|\frac{\partial^2 f_0}{\partial x\,\partial y}\right\|_{L^2(Q)}.$$

We have

$$\int_0^1 \left| \frac{\partial f_0}{\partial x}(x,1) \frac{s(\beta)}{\beta} \frac{s(\alpha x)}{\alpha} \, \mathrm{d}x \right| \leqslant k(\alpha)k(\beta) \int_0^1 \left| \int_0^1 \frac{\partial}{\partial y} \left(y \frac{\partial f_0}{\partial x}(x,y) \right) \mathrm{d}y \right| \mathrm{d}x$$
$$\leqslant k(\alpha)k(\beta) \left[\left\| \frac{\partial f_0}{\partial x} \right\|_{L^2(Q)} + \left\| \frac{\partial^2 f_0}{\partial x \, \partial y} \right\|_{L^2(Q)} \right].$$

Simultaneously

$$\left| \int_0^1 \frac{\partial f_0}{\partial y}(1,y) \frac{s(\beta y)}{\beta} \frac{s(\alpha)}{\alpha} \, \mathrm{d}y \right| \leqslant k(\alpha)k(\beta) \left[\left\| \frac{\partial f_0}{\partial y} \right\|_{L^2(Q)} + \left\| \frac{\partial^2 f_0}{\partial x \partial y} \right\|_{L^2(Q)} \right].$$

Therefore

$$\left| \int_0^1 \int_0^1 f_0(x,y) c(\alpha x) c(\beta y) \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant 4k(\alpha)k(\beta) \| f_0 \|_{H^2(Q)}.$$

It implies that

$$\int_{\mathbb{R}^2 \setminus D_r} \left[\int_Q f_0(x,y) c(\alpha x) c(\beta y) \, \mathrm{d}x \, \mathrm{d}y \right]^2 \mathrm{d}\alpha \, \mathrm{d}\beta$$
$$\leqslant 128 \| f_0 \|^2_{H^2(Q)} \left\{ \int_{|\alpha| \leqslant \frac{r}{\sqrt{2}}} \int_{|\beta| \geqslant \frac{r}{\sqrt{2}}} + \int_{|\alpha| \geqslant \frac{r}{\sqrt{2}}} \int_{|\beta| \geqslant \frac{r}{\sqrt{2}}} (k(\alpha))^2 (k(\beta))^2 \, \mathrm{d}\alpha \, \mathrm{d}\beta \right\}$$
$$\leqslant 128 \| f_0 \|^2_{H^2(Q)} \| k \|^2_{L^2(\mathbb{R})} \int_{|\beta| \geqslant \frac{r}{\sqrt{2}}} \frac{1}{\beta^2} \, \mathrm{d}\beta$$
$$\leqslant \frac{1536}{r} \| f_0 \|^2_{H^2(Q)}.$$

We complete the proof of Lemma 3. □

Proof of Theorem 2. By choosing $r = \frac{1}{\sqrt[8]{\varepsilon}} > 1$ and $D = \{(\alpha, \beta)/\alpha^2 + \beta^2 \leqslant r^2\}$.

By (7), we have

$$\mathrm{e}^{\alpha^2+\beta^2} \int_0^1 \int_0^1 g_0(x,y) c(\alpha x) c(\beta y) \, \mathrm{d}x \, \mathrm{d}y = 2\pi \left[-\int_0^1 \mathrm{e}^{(\alpha^2+\beta^2)t} \varphi_0(t) \, \mathrm{d}t \right] \widehat{\tilde{f}_0}(\alpha, \beta). \tag{17}$$

Therefore

$$\widehat{\tilde{f}_0}(\alpha, \beta) = -\mathrm{e}^{\alpha^2+\beta^2} \frac{\int_0^1 \int_0^1 g_0(x,y) c(\alpha x) c(\beta y) \, \mathrm{d}x \, \mathrm{d}y}{2\pi \int_0^1 \mathrm{e}^{(\alpha^2+\beta^2)t} \varphi_0(t) \, \mathrm{d}t}.$$

Hence,

$$\tilde{f}_0(\xi,\zeta)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\widehat{\tilde{f}_0}(\alpha,\beta)\mathrm{e}^{\mathrm{i}(\alpha\xi+\beta\zeta)}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

$$=-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathrm{e}^{\alpha^2+\beta^2}\frac{\int_0^1\int_0^1 g_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi_0(t)\,\mathrm{d}t}\mathrm{e}^{\mathrm{i}(\alpha\xi+\beta\zeta)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{18}$$

Let

$$\tilde{f}_\varepsilon(\xi,\zeta)\equiv-\frac{1}{4\pi^2}\int_{D}\mathrm{e}^{\alpha^2+\beta^2}\frac{\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t}\mathrm{e}^{\mathrm{i}(\alpha\xi+\beta\zeta)}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{19}$$

We denote $\|.\|_2$, norm in $L^2(\mathbb{R}^2)$.
We have

$$\|\tilde{f}_\varepsilon-\tilde{f}_0\|_2^2=\|\widehat{\tilde{f}}_\varepsilon-\widehat{\tilde{f}}_0\|_2^2=\int_D|\widehat{\tilde{f}}_\varepsilon(\alpha,\beta)-\widehat{\tilde{f}}_0(\alpha,\beta)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta+\int_{\mathbb{R}^2\setminus D}|\widehat{\tilde{f}}_0(\alpha,\beta)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{20}$$

With $(\alpha,\beta)\in D$, we get

$$\begin{aligned}
&2\pi|\widehat{\tilde{f}}_\varepsilon(\alpha,\beta)-\widehat{\tilde{f}}_0(\alpha,\beta)|\\
&=\left|\mathrm{e}^{\alpha^2+\beta^2}\left[\frac{\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t}-\frac{\int_0^1\int_0^1 g_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi_0(t)\,\mathrm{d}t}\right]\right|\\
&=\left|\mathrm{e}^{\alpha^2+\beta^2}\left[\frac{\int_0^1\int_0^1[g_0(x,y)-g(x,y)]c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi_0(t)\,\mathrm{d}t}\right.\right.\\
&\quad+\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\left.\left.\times\left(\frac{1}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi_0(t)\,\mathrm{d}t}-\frac{1}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t}\right)\right]\right|\\
&\leqslant\mathrm{e}^{\alpha^2+\beta^2}\frac{\|g-g_0\|_{L^2(Q)}}{C_0(\mathrm{e}^{\alpha^2+\beta^2}-1)}(\alpha^2+\beta^2)\\
&\quad+\mathrm{e}^{\alpha^2+\beta^2}\|g\|_{L^2(Q)}\left|\frac{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}(\varphi(t)-\varphi_0(t))\,\mathrm{d}t}{\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi_0(t)\,\mathrm{d}t\int_0^1\mathrm{e}^{(\alpha^2+\beta^2)t}\varphi(t)\,\mathrm{d}t}\right|\\
&\leqslant\frac{\varepsilon(\alpha^2+\beta^2)}{C_0(1-\mathrm{e}^{-(\alpha^2+\beta^2)})}+\|g\|_{L^2(Q)}\frac{\sqrt{(\alpha^2+\beta^2)^3}\sqrt{1-\mathrm{e}^{-2(\alpha^2+\beta^2)}}}{\sqrt{2}C_0^2(1-\mathrm{e}^{-(\alpha^2+\beta^2)})^2}\varepsilon.
\end{aligned}$$

For $r \geqslant 1$, using the inequalities,

$$\begin{cases} \dfrac{u}{1-\mathrm{e}^{-u}} \leqslant 2u \leqslant \mathrm{e}u \leqslant \mathrm{e}r^2, & \forall u \geqslant 1 \\ \dfrac{u}{1-\mathrm{e}^{-u}} \leqslant \mathrm{e} \leqslant \mathrm{e}r^2, & \forall u \in [0,1) \end{cases} \qquad u = \alpha^2+\beta^2 \leqslant r^2$$

we have

$$\frac{\varepsilon(\alpha^2+\beta^2)}{C_0(1-\mathrm{e}^{-(\alpha^2+\beta^2)})} \leqslant \frac{\varepsilon \mathrm{e} r^2}{C_0} \leqslant \frac{\varepsilon \mathrm{e}^{3/2} r^3}{C_0}$$

and

$$\|g\|_{L^2(Q)} \frac{\sqrt{(\alpha^2+\beta^2)^3}\sqrt{1+\mathrm{e}^{-(\alpha^2+\beta^2)}}}{C_0^2\sqrt{2}\sqrt{\left(1-\mathrm{e}^{-(\alpha^2+\beta^2)}\right)^3}}\varepsilon \leqslant \|g\|_{L^2(Q)}(\mathrm{e}r^2)^{3/2}\frac{\varepsilon}{C_0^2} \leqslant \frac{\|g\|_{L^2(Q)}\mathrm{e}^{3/2}r^3\varepsilon}{C_0^2}.$$

Therefore,

$$\begin{aligned} 4\pi^2 \int_D |\widehat{\tilde{f}}_\varepsilon(\alpha,\beta) - \widehat{\tilde{f}}_0(\alpha,\beta)|^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta &\leqslant \frac{\varepsilon^2 \mathrm{e}^3 r^6}{C_0^4}[C_0 + \|g\|_{L^2(Q)}]^2 \int_D \mathrm{d}\alpha\,\mathrm{d}\beta \\ &\leqslant \pi \frac{\varepsilon^2 \mathrm{e}^3 r^8}{C_0^4}[C_0 + \|g\|_{L^2(Q)}]^2. \end{aligned} \tag{21}$$

If we put

$$\eta(\varepsilon) \equiv 4\pi^2 \int_{\mathbb{R}^2\setminus D} |\widehat{\tilde{f}}_0(\alpha,\beta)|^2 \,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{22}$$

then $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By (20)–(22), we get

$$4\pi^2 \|\tilde{f}_\varepsilon - \tilde{f}_0\|_2^2 \leqslant \frac{\pi \mathrm{e}^3}{C_0^4}[C_0 + \|g\|_{L^2(Q)}]^2 \varepsilon + \eta(\varepsilon),$$

it means that

$$\|4\tilde{f}_\varepsilon - f_0\|_{L^2(Q)} \leqslant \frac{2}{\pi}\sqrt{\frac{\pi \mathrm{e}^3}{C_0^4}[C_0 + \|g\|_{L^2(Q)}]^2 \varepsilon + \eta(\varepsilon)} \tag{23}$$

Put $f_\varepsilon = 4\tilde{f}_\varepsilon$, we get (9).

Noting that

$$\widehat{\tilde{f}}_0(\alpha,\beta) = \frac{1}{2\pi}\int_0^1\int_0^1 f_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y.$$

By using Lemma 3, if $f_0 \in H^2(Q)$ then we get

$$\|f_\varepsilon - f_0\|_{L^2(Q)} \leqslant \frac{1}{\pi}\sqrt{\frac{\pi e^3}{C_0^4}[C_0 + \|g\|_{L^2(Q)}]^2\varepsilon + \frac{384}{\pi^2}\|f_0\|^2_{H^2(Q)}\varepsilon^{1/8}}.$$

This completes the proof. □

Proof of Theorem 3. Let

$$\Phi(\mu) \equiv \int_0^1 e^{t\mu^2}\varphi(t)\,dt.$$

Putting

$$\mathscr{D}_\varepsilon = \left\{(\alpha, \beta) \Big/ \left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right| \geqslant \varepsilon^a \text{ and } \alpha^2+\beta^2 < r^2(\varepsilon)\right\},$$

$$D_1 = \{(\alpha, \beta)/\alpha^2+\beta^2 < r^2(\varepsilon)\},$$

$$D_2 = \left\{(\alpha, \beta) \Big/ \left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right| < \varepsilon^a\right\}$$

and

$$F_\varepsilon(\alpha, \beta) \equiv \frac{1}{2\pi}\begin{cases} \dfrac{-e^{\alpha^2+\beta^2}\int_0^1\int_0^1 g(x, y)c(\alpha x)c(\beta y)\,dx\,dy}{\Phi\left(\sqrt{\alpha^2+\beta^2}\right)}, & (\alpha, \beta) \in \mathscr{D}_\varepsilon, \\ 0, & (\alpha, \beta) \notin \mathscr{D}_\varepsilon, \end{cases}$$

$$\tilde{f}_\varepsilon(\alpha, \beta) \equiv \frac{1}{2\pi}\int_{D_1} F_\varepsilon(\alpha, \beta)e^{i(\alpha x+\beta y)}\,d\alpha\,d\beta.$$

We get

$$\begin{aligned}\|\tilde{f}_\varepsilon - \tilde{f}_0\|_2^2 &= \|\widehat{\tilde{f}_\varepsilon} - \widehat{\tilde{f}_0}\|_2^2 \\ &= \int_{\mathscr{D}_\varepsilon} |F_\varepsilon(\alpha, \beta) - \hat{\tilde{f}}_0(\alpha, \beta)|^2\,d\alpha\,d\beta + \int_{D_1\cap D_2} |\widehat{\tilde{f}_0}(\alpha, \beta)|^2\,d\alpha\,d\beta \\ &\quad + \int_{\mathbb{R}^2\setminus D_1} |\widehat{\tilde{f}_0}(\alpha, \beta)|^2\,d\alpha\,d\beta \equiv I_1 + I_2 + I_3.\end{aligned}$$

Firstly, estimating I_2, we have

$$\begin{aligned}\left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)-\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right| &\leqslant \int_0^1 \mathrm{e}^{(\alpha^2+\beta^2)t}|\varphi(t)-\varphi_0(t)|\,\mathrm{d}t \\ &\leqslant \|\varphi-\varphi_0\|_{L^2(0,1)}\sqrt{\int_0^1 \mathrm{e}^{2(\alpha^2+\beta^2)t}\,\mathrm{d}t} \\ &\leqslant \varepsilon\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}.\end{aligned}$$

If $(\alpha,\beta)\in D_1\cap D_2$ then

$$\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right| < \varepsilon^a+\varepsilon\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}} < \varepsilon^a+\varepsilon^{1-b}, \tag{24}$$

since $r(\varepsilon)=\sqrt{b\ln\frac{1}{\varepsilon}}$, $0<b<1$. We have

$$|\widehat{\tilde{f}}_0(\alpha,\beta)| = \frac{1}{2\pi}\left|\int_0^1\int_0^1 f_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \frac{1}{2\pi}\|f_0\|_{L^2(Q)}. \tag{25}$$

By (24) and (25), we get

$$I_2 \leqslant \frac{1}{4\pi^2}\|f_0\|^2_{L^2(Q)} m(B_{(\varepsilon^a+\varepsilon^{1-b})}) \leqslant \frac{1}{4\pi^2}\|f_0\|^2_{L^2(Q)} C_0(\varepsilon^a+\varepsilon^{1-b})^{\gamma}. \tag{26}$$

Now, estimating I_1, we have in view of $(\alpha,\beta)\in\mathscr{D}_\varepsilon$

$$\left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right| \geqslant \varepsilon^a$$

hence,

$$\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right| \geqslant \varepsilon^a-\varepsilon\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}} \geqslant \varepsilon^a-\varepsilon^{1-b}>0$$

for $0<b<1-a$.

From $(\alpha, \beta) \in \mathscr{D}_\varepsilon$, we have

$$
\begin{aligned}
&|F_\varepsilon(\alpha,\beta) - \widehat{\tilde{f}}_0(\alpha,\beta)| \\
&\quad = \frac{1}{2\pi}\left| -\mathrm{e}^{\alpha^2+\beta^2}\frac{\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\Phi\left(\sqrt{\alpha^2+\beta^2}\right)} - \int_0^1\int_0^1 f_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad = \frac{1}{2\pi}\left| \mathrm{e}^{\alpha^2+\beta^2}\left[\frac{\int_0^1\int_0^1 g(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\Phi\left(\sqrt{\alpha^2+\beta^2}\right)} - \frac{\int_0^1\int_0^1 g_0(x,y)c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y}{\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)} \right]\right| \\
&\quad \leqslant \frac{1}{2\pi}\mathrm{e}^{\alpha^2+\beta^2}\frac{\|g_0\|_{L^2(Q)}\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right) - \Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right|}{\left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right|\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right|} \\
&\qquad + \frac{1}{2\pi}\mathrm{e}^{\alpha^2+\beta^2}\frac{\|\varphi_0\|_{L^2(0,1)}\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}\left|\int_0^1\int_0^1[g(x,y)-g_0(x,y)]c(\alpha x)c(\beta y)\,\mathrm{d}x\,\mathrm{d}y\right|}{\left|\Phi\left(\sqrt{\alpha^2+\beta^2}\right)\right|\left|\Phi_0\left(\sqrt{\alpha^2+\beta^2}\right)\right|} \\
&\quad \leqslant \frac{1}{2\pi}\mathrm{e}^{\alpha^2+\beta^2}\frac{\varepsilon\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}(\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)})}{\varepsilon^a\left(\varepsilon^a - \varepsilon\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}\right)} \\
&\quad \leqslant \frac{1}{2\pi}\,\mathrm{e}^{\alpha^2+\beta^2}\varepsilon^{1-2a}\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}\,\frac{\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)}}{1-\varepsilon^{1-a}\sqrt{\frac{\mathrm{e}^{2(\alpha^2+\beta^2)}-1}{2(\alpha^2+\beta^2)}}} \\
&\quad \leqslant \frac{1}{2\pi}\,\mathrm{e}^{(r(\varepsilon))^2}\varepsilon^{1-2a}\sqrt{\frac{\mathrm{e}^{2(r(\varepsilon))^2}-1}{2(r(\varepsilon))^2}}\,\frac{\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)}}{1-\varepsilon^{1-a}\sqrt{\frac{\mathrm{e}^{2(r(\varepsilon))^2}-1}{2(r(\varepsilon))^2}}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_1 &\leqslant \frac{1}{2}\mathrm{e}^{2(r(\varepsilon))^2}\varepsilon^{2-4a}\frac{\mathrm{e}^{2(r(\varepsilon))^2}}{2}\,\frac{(\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)})^2}{(1-\varepsilon^{1-a}\mathrm{e}^{(r(\varepsilon))^2})^2} \\
&\leqslant \frac{1}{4}\,\mathrm{e}^{2b\ln(1/\varepsilon)}\varepsilon^{2-4a}\mathrm{e}^{2b\ln(1/\varepsilon)}\frac{(\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)})^2}{(1-\varepsilon^{1-a}\mathrm{e}^{b\ln(1/\varepsilon)})^2} \\
&\leqslant \frac{1}{4}\,\frac{1}{\varepsilon^{2b}}\varepsilon^{2-4a}\frac{1}{\varepsilon^{2b}}\frac{(\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)})^2}{\left(1-\varepsilon^{1-a}\frac{1}{\varepsilon^b}\right)^2} \\
&\leqslant \frac{1}{4}\varepsilon^{2-4a-4b}\frac{(\|g_0\|_{L^2(Q)}+\|\varphi_0\|_{L^2(0,1)})^2}{(1-\varepsilon^{1-a-b})^2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|4\tilde{f}_\varepsilon - \tilde{f}_0\|_2^2 \leqslant & \frac{1}{4}\varepsilon^{2-4a-4b}\frac{(\|g_0\|_{L^2(Q)} + \|\varphi_0\|_{L^2(0,1)})^2}{(1-\varepsilon^{1-a-b})^2} \\
& + \frac{1}{4\pi^2}\|f_0\|_{L^2(Q)}^2 C_0(\varepsilon^a + \varepsilon^{1-b})^\gamma + \eta_b(\varepsilon),
\end{aligned}
$$

where $I_3 = \eta_b(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Putting $f_\varepsilon = 4\tilde{f}_\varepsilon$, we have

$$
\begin{aligned}
\|f_\varepsilon - f_0\|_{L^2(Q)}^2 \leqslant & 4\frac{1}{4}\varepsilon^{2-4a-4b}\frac{(\|g_0\|_{L^2(Q)} + \|\varphi_0\|_{L^2(0,1)})^2}{(1-\varepsilon^{1-a-b})^2} \\
& + \frac{1}{4\pi^2}\|f_0\|_{L^2(Q)}^2 C_0(\varepsilon^a + \varepsilon^{1-b})^\gamma + \eta_b(\varepsilon). \qquad (27)
\end{aligned}
$$

Choose $b \in \left(\max\left\{0, \frac{1-4a}{4}\right\}, \frac{1-2a}{2}\right)$, $a > 0$; that implies $0 < 2-4a-4b < 1$, $a+b < 1$, $1-b > 0$, $0 < a < \frac{1}{2}$ and therefore from (27) we can deduce that there exist $C_b > 0$, $\gamma_b > 0$ independent of g_0, f_0, φ_0 and a function $\eta_b(\varepsilon)$ such that $\lim_{\varepsilon\downarrow 0}\eta_b(\varepsilon) = 0$ and that

$$
\|f_\varepsilon - f_0\|_{L^2(Q)} \leqslant \sqrt{C_b\varepsilon^{\gamma_b} + \eta_b(\varepsilon)}.
$$

By using Lemma 3, if $f_0 \in H^2(Q)$ then we have

$$
\|f_\varepsilon - f_0\|_{L^2(Q)} \leqslant \sqrt{C_b\varepsilon^{\gamma_b} + \frac{384}{\pi^2}\|f_0\|_{H^2(Q)}^2\left(b\ln\frac{1}{\varepsilon}\right)^{-1/2}}.
$$

We complete the proof of Theorem 3. □

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